MAC-CPTM Situations Project

Situation 65: Square root of i

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Prompt

Knowing that a Computer Algebra System (CAS) had commands such as **cfactor** and **csolve** to factor and solve complex numbers respectively, a teacher was curious about what would happen if she entered $\sqrt{i}$. The result was $\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$. Why would a CAS give a result like this?

Commentary

When using a CAS, students and teachers can encounter situations that cause them to question why the CAS may give a particular result. Symbolic verification and manipulation can be used to confirm results given by a CAS. Mathematical focus 1 accounts for the reasoning behind the symbolic work by confirming that the result makes sense. However, this focus does not deal with how $\sqrt{i}=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$ can make sense within a larger system. To address the underlying mathematical logic relating to why $\sqrt{i}=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$, mathematical foci 2, 3 and 4 utilize representations of complex numbers on the complex plane. Mathematical focus 2 connects powers of i to points of the unit circle on the complex plane and their images under rotations, and mathematical focus 3 uses Euler's formula to represent complex numbers in exponential and trigonometric form. Mathematical focus 4 considers the powers of *i* as elements of cyclic groups.

Mathematical Foci

Mathematical Focus 1

Solving the equation $x^2 = i$, where $x = a + bi$, and verifying the solution to the equation provides a representation of the square root of the imaginary number.

Knowing that any complex number is of the form $a+bi$, where a and b are real numbers, we can determine square roots of i by solving the equation $(a+bi)^2 = i$ for a and b. To solve the equation, first we expand $(a+bi)^2$, and the equation becomes $a^2 + 2abi - b^2 = i$. Equating the real and complex parts of the equation, $a^2 - b^2 = 0$ and $2ab = 1$. Therefore, $a = \pm b$ and either $2b^2 = 1$ or $-2b^2 = 1$. However, since we know that both a and b are real and that $-2b^2 = 1$ has no real solutions, we only consider the equation $2b^2 = 1$. However, if $a = -b$, then $2 \cdot a \cdot b = 2 \cdot -b \cdot b = -2b^2 = 1$, which is not possible, meaning that $a = -b$ is not possible, leaving $a = b$ as the only possibility. Solving $2b^2 = 1$ for b gives $b = \frac{\sqrt{2}}{2}$ and $b = -\frac{\sqrt{2}}{2}$. Therefore the equation $(a+bi)^2 = i$ has two sets of solutions, namely $a = \frac{\sqrt{2}}{2}$, $b = \frac{\sqrt{2}}{2}$ and $a = -\frac{\sqrt{2}}{2}$, $b = -\frac{\sqrt{2}}{2}$. In this way, $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ and $-\frac{\sqrt{2}}{2} + -\frac{\sqrt{2}}{2}i$ are both square roots of i. (See Spencer, 1999.)

One way to verify that a complex number is a square root of another number is to square that complex number and verify that the square and the other number are equivalent. By squaring the expression $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$, we can verify that $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ is a square root of $x^2 = i$. It is useful to note that the symbolic manipulations needed to expand the expression $\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^2$ will treat it as though it were an algebraic expression of the form $(a+b)^2$ from the real domain.

Expanding $\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^2$ gives $\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^2 = \left(\frac{\sqrt{2}}{2}\right)^2 + 2\left(\frac{1}{2}\right)i + \left(\frac{\sqrt{2}}{2}i\right)^2$.

Simplifying $\left(\frac{\sqrt{2}}{2}\right)^2 + 2\left(\frac{1}{2}\right)i + \left(\frac{\sqrt{2}}{2}i\right)^2$ gives $\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^2 = \frac{1}{2} + i - \frac{1}{2} = i$.

Since $\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^2 = i$, we conclude that a square root of $x^2 = i$ is equal to $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$.

Mathematical Focus 2

Relating powers of i to rotations involving the unit circle on the complex plane

Consider the unit circle on the complex plane, and on this circle, consider the point representations of i^0 and i (figure 1).

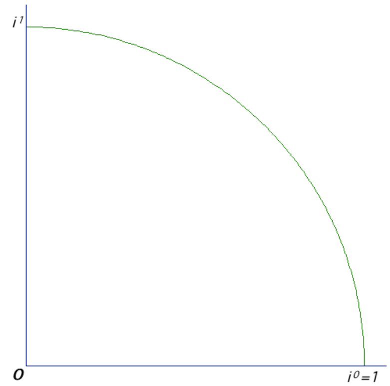

Figure 1 First quadrant of the unit circle on the complex plane.

Note that the point representing i is the image of the point representing i^0 under $\rho_{(O,90°)}$, a rotation of 90° about the origin (***O***). Thus, if the point for i^0 could be represented as (1,0), and if the point for i could be represented as $\rho_{(O,90°)}((1,0))=(0,1)$, then the point for $\sqrt{i}$ can be thought of as the image of the point for i^0 under $\rho_{(O,45°)}$, a rotation of 45° (figure 2). Moreover, the point for i can also be thought about as the image of the point for $\sqrt{i}$ under $\rho_{(O,45°)}$ So, $\rho_{(O,45°)}$ composed with itself is the same as $\rho_{(O,90°)}$. That is, $\rho^2_{(O,45°)}.=\rho_{(O,90°)}$

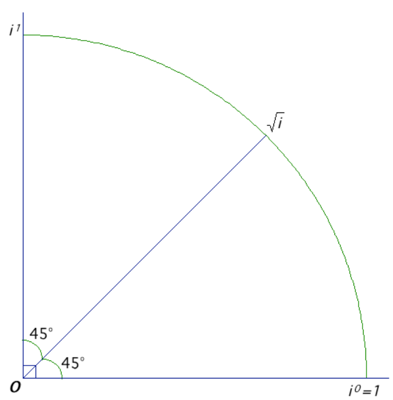

Figure 2 Images of points representing powers of i as rotations.

We can notice that each point on the circle corresponds to the complex number, $\cos x + i\sin x$. This is shown in figure 3.

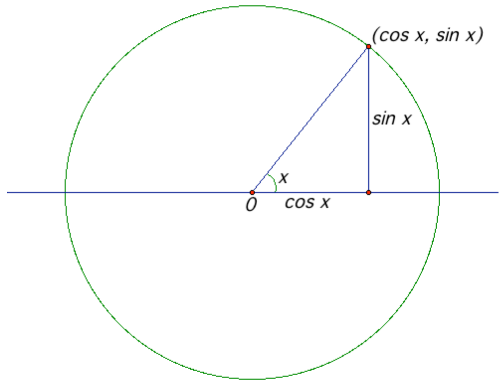

Figure 3 Coordinates of Points on the Complex Unit Circle

The point representing $i^{\frac{1}{2}}$ is the image of the point for i^0 under a rotation of 45^0 about the origin. Therefore, the coordinates of $i^{\frac{1}{2}}$ have to be $(\cos 45^\circ, \sin 45^\circ) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. Thus, $i^{\frac{1}{2}} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$.

Mathematical Focus 3

By using Euler's formula, the connection between the trigonometric representation of any complex number and the square root of the imaginary number is made more explicit.

Knowing that every point on the unit circle on the complex plane corresponds to a complex number z, where $z = \cos\theta + i\sin\theta$, Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$, can be used to express those complex numbers in the exponential form $z = e^{i\theta}$. For example, if we let $\theta = \pi$, we arrive at $z = e^{i\pi} = \cos\pi + i\sin\pi = -1$, which can be represented by the point (-1,0) on the unit circle on the complex plane. Similarly, if we let $\theta = \frac{\pi}{2}$, we arrive at $z = e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$, which can be represented by the point (0,1) on the unit circle on the complex plane. Since we are interested in determining $\sqrt{i}$ and since $e^{i\frac{\pi}{2}} = i$, by Euler's formula, it follows that $\sqrt{e^{i\frac{\pi}{2}}} = \sqrt{i}$. Since $i^{\frac{\pi}{2}} = \left(e^{i\frac{\pi}{2}}\right)^{\frac{1}{2}}$, using properties of exponents, we can conclude that $i^{\frac{1}{2}} = e^{i\frac{\pi}{4}}$. In this way, if we let $\theta = \frac{\pi}{4}$, we arrive at $z = e^{i\frac{\pi}{4}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$, which can be represented by the point $\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$ on the unit circle on the complex plane. Since $e^{i\frac{\pi}{4}}=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$, and $i^{\frac{1}{2}}=e^{i\frac{\pi}{4}}$, we can conclude that $i^{\frac{1}{2}}=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$.

Mathematical Focus 4

The value of the square root of the imaginary number can be determined by investigating this value in relationship to cyclic groups.

This situation deals with $\sqrt{i}$, which can of course be written as $i^{\frac{1}{2}}$. So, one way to go about this situation is to look for patterns in the powers of *i*. To begin with, let's look at the integer powers of *i*, starting with i^0. If we were to plot points representing the imaginary numbers i^0, i^1, i^2, i^3, we obtain the following figure (figure 4).

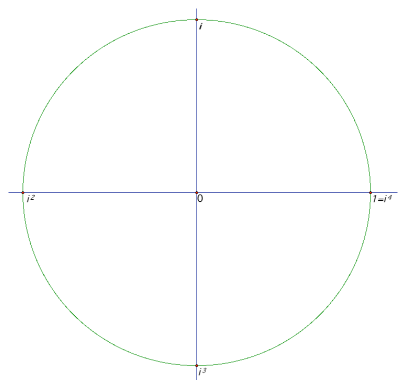

Figure 4 The four powers of *i*

Note that all four powers of *i* above are on the complex unit circle. Moreover, the four points are positioned at equal increments around the circle (exactly at 90° increments). Furthermore, we can see the that fourth power of *i* can be plotted in the same position as the zero power of *i* (i.e., $i^4=i^0=1$). We can also see that every integer power of *i* greater than 3 is plotted on the above four points around the complex unit circle. This arrangement, at equal increments, around a circle of the four powers of *i*, and the cyclic property of the powers described above, leads to looking at a cyclic group generated by *i*.

Consider a cyclic group, $(G,\circ)$, of order 4, isomorphic to $(\mathbf{Z}_4,+)$, which can be generated by using the imaginary number *i* as the generator, i.e. $i^{4k}=1$, where $k\geq 0$ and *k* is an integer. Note that 1 is called the identity element of the group *G*. Also, we can list all the elements of this group by considering the powers of *i*, i.e. $G=\langle i\rangle_4=\{1,i^1,i^2,i^3\}$. As discussed before, the elements of the cyclic group, *G*, can be very naturally illustrated as four symmetric points on the unit circle in the complex plane, as shown in figure 4.

Since we are interested in $i^{\frac{1}{2}}$, however, we can further this discussion of the powers of *i*, by examining the first eight powers of *i*, increasing the powers in increments of $\frac{1}{2}$. Thus, now we are increasing the order of the group from 4 to 8. So now we have the cyclic group, $(H, \circ)$, isomorphic to $(\mathbf{Z}_8, +)$, given by $H = \langle i \rangle_8 = \{1, i^{\frac{1}{2}}, i^1, i^{\frac{3}{2}}, i^2, i^{\frac{5}{2}}, i^3, i^{\frac{7}{2}}\}$. This group can also be illustrated on the unit circle on the complex plane as shown in figure 5.

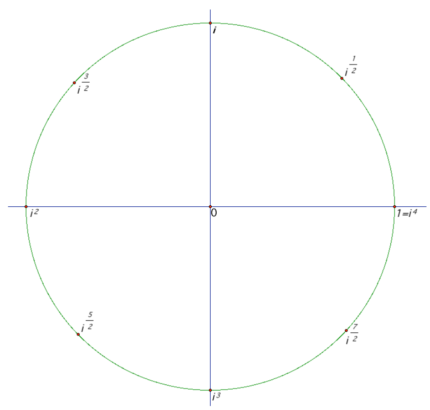

Figure 5 The Cyclic Group $(H, \circ)$

To obtain the co-ordinates of these points on the complex unit circle, we can refer back to focus 2 and obtain $i^{1/2} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$.

References

Spencer, P. (1999, April 19), *Question Corner—What is the Square Root of i?* Retrieved September 14, 2006 from University of Toronto, Mathematics Network Web site: http://www.math.toronto.edu/mathnet/questionCorner/rootofi.html